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Overrings of Prüfer Domains

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Let D be an integral domain with identity having quotient field K. By an *overring* of D we mean any domain between D and K. By a *quotient ring* of D we mean an overring of D of the form D_N for some nonempty multiplicative system N contained in $D - \{0\}$. We say D_N is a *prime quotient ring* of D if $N = D - P$ for some proper prime ideal P of D and, following the notation of [*10*, p. 228], we write $D_P = D_N$ in this case. We say that D has the *QR-property* if each overring of D is a quotient ring of D [*4*]. If for each maximal ideal M of D, D_M is a rank-one discrete valuation ring, then D is *almost Dedekind*. If Δ is the set of maximal ideals of D, we say D has *property* (#) if for Δ_1 and Δ_2 distinct subsets of Δ we have $\bigcap_{P\in\Delta_1} D_P \neq \bigcap_{P\in\Delta_2} D_P$. In [*3*], it was conjectured that an almost-Dedekind domain need not have property (#). The validity of this conjecture is established here by Theorem 3, which states that an almost-Dedekind domain satisfying property (#) is a Dedekind domain. In Section 1 we consider property (#) in an arbitrary integral domain D with identity. In Section 2 we consider the case in which D is a Prüfer domain. The examples of Section 3 show that the results obtained are, in most cases, the best possible.

1. Preliminary Results on Property (#)

In this section D denotes an integral domain with identity having quotient field K and Δ denotes the set of maximal ideals of D. We consider consequences of property (#) on D.

LEMMA 1. *D has property (#) if and only if for $P \in \Delta$ and $\Delta_P = \Delta - \{P\}$, $\bigcap_{M\in\Delta_P} D_M \nsubseteq D_P$.*

Proof. The condition is obviously necessary in order that D have property (#). And if the condition holds, let Δ_1 and Δ_2 be distinct subsets of Δ, say,

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$P \in \Delta_1 - \Delta_2$. Then $D_P \supseteq \bigcap_{M\in\Delta_1} D_M$, but $D_P \not\supseteq \bigcap_{M\in\Delta_P} D_M \subseteq \bigcap_{M\in\Delta_2} D_M$. Consequently $\bigcap_{M\in\Delta_1} D_M \not\supseteq \bigcap_{M\in\Delta_2} D_M$. In particular, $\bigcap_{M\in\Delta_1} D_M \neq \bigcap_{M\in\Delta_2} D_M$ and D has property (#).

LEMMA 2. *If for $P \in \Delta$, $P \nsubseteq \bigcup_{M\in\Delta_P} M$, then D has property (#).*

Proof. If $p \in P - \bigcup_{M\in\Delta_P} M$, the $1/p \in (\bigcap_{M\in\Delta_P} D_M) - D_P$. Thus D has property (#) by Lemma 1.

For D one-dimensional, the condition that $P \nsubseteq \bigcap_{M\in\Delta_P} M$ is equivalent to P's being the radical of a principal ideal, or to the condition that there is a P-primary ideal of D which is principal. Thus by taking D to be a Dedekind domain whose class group is not a torsion group [cf. *4*, p. 102; *9*, p. 146], we see that the conditions of Lemma 2 are not necessary in order that D have property (#). But for D having the QR-property they are necessary as is shown by the following result.

LEMMA 3. *If D has the QR-property, if $\{P_\alpha\}$ is a set of proper prime ideals of D and if P is a proper prime distinct from each P_α, the statements "$P \subseteq \bigcup P_\alpha$" and "$D_P \supseteq \bigcap D_{P_\alpha}$" are equivalent.*

Proof. In the proof of Lemma 2 we have shown that if $D_P \supseteq \bigcap D_{P_\alpha}$ then $P \subseteq \bigcup P_\alpha$. The converse follows immediately from Proposition 1.2 of [4].

COROLLARY 1. *If D has the QR-property and is one-dimensional, then D has property (#) if and only if each maximal ideal of D is the radical of a principal ideal.*

Corollary 1 shows that for a wide class of almost-Dedekind domains J, J has property (#) if and only if J is Dedekind. This result will be proved in general by Theorem 3. Namely, let J be the integral closure of Z in an infinite algebraic number field K. K can be expressed as the union of an ascending sequence of finite algebraic number fields; $K = \bigcup K_i$, and $J = \bigcup Z_i$ where Z_i, the integral closure of Z in K_i, is well known to be a Dedekind domain with a finite class group, hence a domain with the QR-property [*2*, p. 200], [*4*, p. 100]. Thus J is one-dimensional and has the QR-property [*4*, p. 99]. Consequently, if J is almost Dedekind and has property (#) then Corollary 1 shows that if M is maximal in J, (m) is M-primary for some $m \in M$. By Theorem 1 of [*3*], $(m) = M^k$ for some positive integer k so that M is invertible. Then by a well-known result of Nakano [*7*], J is Dedekind.

Remark. By Corollary 1.4 of [*1*], J obtained as above is almost Dedekind if and only if no maximal ideal of J is idempotent. In [*8*], Nakano gives necessary and sufficient conditions on the sequence $\{K_i\}$ in order that J contain no indempotent maximal ideals.

2. Prüfer Domains and Property (#)

Our notation in this section is as in the first section except that here we always require that D *be a Prüfer domain*. The principal result of the section is Theorem 3 which shows that if D is almost Dedekind and has property (#) then D is Dedekind. Our first result is a generalization to Prüfer domains of Theorem 4 of [*3*]. Only part (i) of Theorem 1 is used in the remainder of this paper.

Theorem 1. *Suppose D' is an overring of D, and let Ω be the set of prime ideals P of D such that $PD' \subset D'$. Then*

(i) *If M is a maximal ideal of D' and if $P = M \cap D$, then $D_P = D_M'$ and $M = PD_P \cap D'$. Therefore D' is Prüfer.*

(ii) *For P a proper prime ideal of D, $P \in \Omega$ if and only if $D_P \supseteq D'$. Further, $D' = \bigcap_{P\in\Omega} D_P$.*

(iii) *If A' is an ideal of D' and if $A = A' \cap D$, then $A' = AD'$.*

(iv) *$\{PD'\}_{P\in\Omega}$ is the set of proper prime ideals of D'.*

Proof. In (i) we have $D_M' = D'_{D'-M} \supseteq D'_{D-P} \supseteq D_{D-P} = D_P$. Since D_P is a valuation ring, D_M' is also a valuation ring and in fact $D_M' = (D_P)_Q$ for some prime ideal Q of D_P. Hence $Q = P_\alpha D_P$ for some prime ideal $P_\alpha \subseteq P$ and $D_M' = (D_P)_{P_\alpha D_P} = D_{P_\alpha}$ [*10*, pp. 223–231]. Therefore $MD_M' = P_\alpha D_{P_\alpha}$ and $P_\alpha = P_\alpha D_{P_\alpha} \cap D = MD_M' \cap D = (MD_M' \cap D') \cap D = M \cap D' = P$. Thus $D_M' = D_P$ and $M = MD_M' \cap D' = PD_P \cap D'$ as asserted.

The proof of (ii) follows by a slight modification of the proof of part (c) of Theorem 4 of [*3*]. Also the proof in [*3*] shows that (iv) is valid as soon as (ii) and (iii) hold. Hence to complete the proof we establish (iii). We may assume $(0) \subset A' \subset D'$. Obviously $AD' \subseteq A'$. Now $A' = \bigcap (A'D'_{M_\alpha} \cap D')$, M_α running over all maximal ideals of D' [*11*, p. 94]. If $P_\alpha = M_\alpha \cap D$ we have $D'_{M_\alpha} = D_{P_\alpha}$ by (i). Hence if $x \in A'D'_{M_\alpha} = A'D_{P_\alpha}$, then $x = a'/v$ for some $a' \in A'$, $v \in D - P_\alpha$. But $A' \subseteq D_{P_\alpha}$ so $a' = a/u$ for some $a \in D$, $u \in D - P_\alpha$. We then have $a = a'u \in A' \cap D = A$ and $x = a/uv \in AD_{P_\alpha} = AD'D_{P_\alpha} = AD'D'_{M_\alpha}$. We conclude that $A'D'_{M_\alpha} = AD'D'_{M_\alpha}$ for each α so that $A' = \bigcap (A'D'_{M_\alpha} \cap D') = \bigcap (AD'D'_{M_\alpha} \cap D') = AD'$.

Corollary 2. *Suppose D is one-dimensional. If D has property (#), so does each overring of D.*

Proof. Let D' be an overring of D, let M be maximal in D' and let $\{M_\alpha\}$ be the set of maximal ideals distinct from M. If $P = M \cap D$ and $P_\alpha = M_\alpha \cap D$, then $\{P, P_\alpha\} \subseteq \Delta$ since D is one-dimensional. Theorem 1 shows that the ideals P, P_α, P_β are distinct for $\alpha \neq \beta$, that $D_M' = D_P$, and that $D'_{M_\alpha} = D_{P_\alpha}$

for all α. Since D has property (#) we then have $D_{M}' = D_P \nsupseteq \bigcap D_{P_\alpha} = \bigcap D'_{M_\alpha}$.

LEMMA 4. *Let D^* be the integral closure of D in L, an algebraic extension of K. Then*

(a) *D^* is Prüfer.*

(b) *if P is prime in D, if P^* is prime in D^*, and if v and v^* are the valuations of K and L, respectively, associated with the valuation rings D_P and $D^*_{P^*}$, respectively, the concepts "v extends v^*" and "P^* lies over P" are equivalent.*

(c) *if L is of finite degree over K, there are only finitely many primes of D^* lying over a given prime P of D.*

Proof. (a) is proved by Krull in [5, p. 555]; (b) follows from Lemma 1 of [11, p. 24]; then (c) follows from (b) and Corollary 4 [11, p. 27].

LEMMA 5. *Let v be a valuation of a field L_0 with valuation ring D_v and let $\{L_i\}_{i=0}^{\infty}$ be an ascending sequence of finite algebraic extensions of L_0. Let $F = \bigcup_{i=0}^{\infty} L_i$, let $\mathscr{S} = \{v_\lambda\}_{\lambda\in\Lambda}$ be the set of valuations of F which are extensions of v, and let F_λ be the valuation ring associated with v_λ. In order that for each $\lambda \in \Lambda$ we have $F_\lambda \nsupseteq \bigcap_{u\neq\lambda} F_u$, it is necessary and sufficient that $\mathscr{S}$ be finite.*

Proof. If $\mathscr{S} = \{V_1, V_2, \ldots, V_n\}$ is finite, then because F is algebraic over L_0, $\mathscr{S}$ is an independent set of valuations. That $F_i \nsupseteq \bigcap_{j\neq i} F_j$ for each i then follows from the approximation theorem for independent valuations [11, p. 41].

Now suppose $\mathscr{S}$ is not a finite set. v has only finitely many extensions $w_{11}, \ldots, w_{1k_1}$ to L_1 and for $\lambda \in \Lambda$, v_λ is an extension of some w_{ij}. Hence there exists an extension w_1 of v to L_1 such that w_1 has infinitely many extensions to F. By induction we obtain a sequence $\{w_i\}_{i=1}^{\infty}$ such that for each i, w_i is a valuation on L_i having infinitely many extensions to F and such that for $i \leqslant j$, w_j is an extension of w_i to L_j. The sequence $\{w_i\}_{i=1}^{\infty}$ then defines a unique valuation w on F such that w_i is the restriction of w to L_i for all i. We show that $F_w \supseteq \bigcap_{v_\lambda \neq w} F_\lambda$. Thus let $x \in F - F_w$. For some integer i we then have $x \in L_i - F_w$; hence $w_i(x) < 0$. By assumption there exists an extension v_λ of w_i to F distinct from w. Hence $v_\lambda(x) = w_i(x) < 0$ so $x \notin \bigcap_{v_\lambda \neq w} F_\lambda$, implying our desired conclusion.

COROLLARY 3. *Suppose D is one-dimensional and has property (#). Let J be the integral closure of D in L, an algebraic extension of K which may be expressed as an ascending union of finite algebraic extensions of K. Then J has property (#) if and only if for each maximal ideal M of D, there are only finitely many maximal ideals of J lying over M.*

Proof. By Lemma 4, J is one-dimentional and Prüfer. Hence if M is a maximal ideal of J there is associated with J_M a valuation w_M of L. Similarly, for P maximal in D there is a valuation v_P of K associated with D_P. By Lemma 4, $M \cap D = P$ if and only if w_M extends v_P. Lemma 5 then shows that if J has property (#) each v_P must have only finitely many extensions to L. Hence for P maximal in D, there can be only finitely many maximal ideals of J lying over P.

Conversely, if each maximal ideal of D lies under only finitely many maximal ideals of J, let M be maximal in J and let $M_1, ..., M_n$ be the other maximal ideals of J lying over $P = M \cap D$. If $\{M_\alpha\}$ is the set of all other maximal ideals of J and if $\{P_\beta\}$ is the set of maximal ideals of D distinct from P, then $D_P \nsupseteq \bigcap D_{P_\beta}$ by hypothesis on D. Hence if $x \in (\bigcap D_{P_\beta}) - D_P$, then $x \in J_M - (\bigcap J_{M_\alpha})$. Now $M \nsupseteq \bigcap_{i=1}^n M_i$ so we choose $t \in (\bigcap_{i=1}^n M_i) - M$. Since w_{M_i} has rank one for each i, there exists a positive integer k such that $w_{M_i}(t^k) \geqslant - w_{M_i}(x)$ for each i. For such a k we then have $t^k x \in (\bigcap J_{M_i}) \cap (\bigcap J_{M_\alpha})$, $t^k x \notin J_M$. Consequently, J has property (#).

LEMMA 6. *Suppose D is one-dimensional. A sufficient condition in order that D have property (#) is that each maximal ideal of D contain an element which is contained in only finitely many maximal ideals of D.*

Proof. The proof is quite similar to that of the converse of Corollary 3. Let M be a maximal ideal of D and let $\{M_\alpha\} = \Delta - \{M\}$. Let v be the valuation associated with D_M and v_α the valuation associated with D_{M_α}. Let m be a nonzero element of M which is contained in only finitely many maximal ideals of D distinct from M. Let this set be $\{M_i\}_{i=1}^n$. Then for $t \in (\bigcap_{i=1}^n M_i) - M$ and for a suitably chosen positive integer k we have $v_\alpha(t^k/m) \geqslant 0$ for each α while $v(t^k/m) = -v(m) < 0$. Hence $t^k/m \in (\bigcap D_{M_\alpha}) - D_M$ and D has property (#) as asserted.

Note. Example 1 of Section 3 shows that a one-dimensional Prüfer domain in which *each nonunit* is contained in only finitely many maximal ideals need not be almost Dedekind.

Before proving Theorem 2 we establish the following lemma.

LEMMA 7. *Suppose D is one-dimensional and that the Jacobson radical of D is nonzero. If $\Delta = \{M_\beta\}$ and if $M_\alpha \in \Delta$ is the radical of an ideal with two generators, then there exists $m_\alpha \in M_\alpha$ such that $1 - m_\alpha \in M_\beta$ for each $\beta \neq \alpha$.*

Proof. Let x be a nonzero element of the Jacobson radical of D. If v_β is the valuation associated with D_{M_β} for each β, then $v_\beta(x) > 0$ for each β. By hypothesis, there exist $u, t \in M_\alpha$ such that $M_\alpha = \sqrt{(u, t)}$. Since v_α has rank one, there is an integer n such that $v_\alpha(u^n) > v_\alpha(x)$, $v_\alpha(t^n) > v_\alpha(x)$.

Then if $B = (t^n, u^n, x)$, $\sqrt{B} = M_\alpha$ and the minimum v_α-value of an element of B is $v_\alpha(x)$. Now $x \in B$ and B is invertible since D is Prüfer. Thus $(x) = AB$ for some ideal A of D. A is also invertible so that v_α attains its minimal value on A, which in this case must be zero because of our observation regarding the v_α-values of elements of B. That is, $A \nsubseteq M_\alpha$. Yet for $\beta \neq \alpha$, $AB = (x) \subseteq M_\beta$ while $B \nsubseteq M_\beta$. It follows that $A \subseteq \bigcap_{\beta \neq \alpha} M_\beta$. We choose $a \in A - M_\alpha$. Since M_α is maximal $m_\alpha + da = 1$ for some $m_\alpha \in M_\alpha$, $d \in D$. Then for $\beta \neq \alpha$, $1 - m_\alpha = da \in M_\beta$.

LEMMA 8. *Let R be a commutative ring with identity e and let $\sum = \{M_\lambda\}_{\lambda \in \Lambda}$ be the set of maximal ideals of R. If for each $M_\lambda \in \sum$ there exists $m_\lambda \in M_\lambda$ such that $e - m_\lambda \in \bigcap_{\mu \neq \lambda} M_\mu$, then Λ is a finite set.*

Proof. Suppose Λ is not finite. Then there exists a well-ordering $<$ of Λ under which Λ has no largest element. Then for $\lambda \in \Lambda$ we define $A_\lambda = \bigcap_{\beta > \lambda} M_\beta$. By hypothesis, $e - m_\lambda \in A_\lambda - M_\lambda$ for each λ. Hence $\{A_\lambda\}_{\lambda \in \Lambda}$ is a chain of proper ideals of R. Then $A = \bigcup A_\lambda$ is again a proper ideal of R since $e \notin A$. But by choice of A_λ, A is not contained in any maximal ideal of R, a contradiction. Hence Λ is finite as asserted.

THEOREM 2. *If D is one-dimensional and if $\Delta = \{M_\beta\}$, then given $M_\alpha \in \Delta$, these statements are equivalent.*

(a) $D_{M_\alpha} \nsupseteq \bigcap_{\beta \neq \alpha} D_{M_\beta}$.

(b) *M_α is the radical of an ideal with two generators.*

(c) *M_α is the radical of a finitely generated ideal.*

Proof. $(a) \to (b)$: Let $\Delta' = \{M_\lambda\} = \Delta - \{M_\alpha\}$, let v_β be the valuation associated with D_{M_β} for each β. Since $D_{M_\alpha} \nsupseteq \bigcap D_{M_\lambda}$, there exist $a, b \in D$ such that $v_\alpha(a) < v_\alpha(b)$ and $v_\lambda(a) \geqslant v_\lambda(b)$ for each λ. Hence $v_\alpha(b/a) > 0$ and $b/a \in M_\alpha D_M$; say $b/a = s/t$ where $s \in M_\alpha$, $t \in D - M_\alpha$. Then $v_\alpha(t) = 0 < v_\alpha(s)$ and $v_\lambda(t) \geqslant v_\lambda(s)$ for each λ. We now let Ω' be the set of M_λ's which contain s and we let $\Omega = \Omega' \cup \{M_\alpha\}$. Ω is the set of maximal ideals of D containing s. We note that if $P \in \Delta$ and if $P \subseteq \bigcup_{T \in \Omega} T$, then $P \in \Omega$. For if $P \notin \Omega$—that is, if $s \notin P$, then $p + ds = 1$ for some $p \in P$, $d \in D$. It then follows that $p \notin T$ for $T \in \Omega$. Whence $p \in P - (\bigcup_{T \in \Omega} T)$. This observation shows that if $N = D - (\bigcup_{T \in \Omega} T)$ and if $D' = D_N$, then $\{TD'\}_{T \in \Omega}$ is the set of maximal ideals of D'. D' is one-dimensional Prüfer by Theorem 1 and $D'_{M_\alpha D'} = D_{M_\alpha} \nsupseteq \bigcap_{T \in \Omega'} D'_{TD'} = \bigcap_{T \in \Omega'} D_T$ by hypothesis. Also if $M_\beta \in \Omega$, $D_{M_\beta} = D'_{M_\beta D'}$ implies v_β is the valuation associated with $D'_{M_\beta D'}$. For each such $M_\beta \in \Omega'$ we then have $v_\beta(t) \geqslant v_\beta(s) > 0$ while $v_\alpha(s) > v_\alpha(t) = 0$. This then implies, as in the proof of Lemma 7 and as previously shown in this proof, that there exists $u \in M_\alpha D' - (\bigcup_{T \in \Omega'} TD')$. There is no loss of generality in assuming $u \in M_\alpha$. We now show that the ideal $B = (s, u)$ in D has radical M_α. Clearly $B \subseteq M_\alpha$. If $M_\lambda \in \Delta' - \Omega'$, then $s \notin M_\lambda$ so $B \nsubseteq M_\lambda$. Moreover, if $M_\lambda \in \Omega'$, then $u \notin M_\lambda D'$, implying $u \notin M_\lambda$, again implying $B \nsubseteq M_\lambda$. Therefore $\sqrt{B} = M_\alpha$, and (b) holds.

Obviously $(b) \to (c)$.

$(c) \to (b)$: We suppose $M_\alpha = \sqrt{B}$ where $B = (b_1, b_2, \ldots, b_n)$. Since B is invertible, $B \supset BM_\alpha$. Thus if $b \in B - BM_\alpha$ then $B^2 + (b) = [B^2 + (b)]B^{-1}B = [B + (b)B^{-1}]B$. Hence $[B^2 + (b)] : B \supseteq B + B^{-1}(b)$ and since $b \notin M_\alpha B$, $B^{-1}(b) \nsubseteq M_\alpha$. Then $\sqrt{B + B^{-1}(b)} \supset M_\alpha$; therefore $B + B^{-1}(b) = [B^2 + (b)] : B = D$, and $B = B^2 + (b)$. Hence for each b_i, there exist $a_{ij} \in B$, $r_i \in D$ such that

$$b_i = \sum_{j=1}^{n} a_{ij} b_j + r_i d$$

or such that $\sum_{j=1}^{n} (\delta_{ij} - a_{ij}) b_j = r_i d$.

If $\| \delta_{ij} - a_{ij} \| = u$ is the determinant of this system, then from Cramer's rule we have $ub_j \in (d)$ for each j. But u is of the form $1 - t$ for some $t \in B$ so that we have $b_j - b_j t \in (d)$ for each j. Consequently, $B = (t, d)$ and (b) holds.

$(b) \to (a)$: We suppose $M_\alpha = \sqrt{(r, s)}$ and we fix $x \in M_\alpha$, $x \neq 0$. If $\{M_\alpha\}$ is the set of maximal ideals of D which contain x, if $N = D - (\bigcup M_\alpha)$, and if $D' = D_N$, D' is one-dimensional Prüfer and $\{M_\alpha D'\}$ is the set of maximal ideals of D'. Hence the Jacobson radical of D' contains the nonzero element x and $M_\alpha D' = \sqrt{(r, s)D'}$. Lemma 8 then shows that there exists $t_\alpha \in M_\alpha$, $n \in N$ such that $t_\alpha/n \in M_\alpha D'$, $(n - t_\alpha)/n = 1 - (t_\alpha/n) \in M_r D'$ for each $r \neq \alpha$. Thus $t_\alpha \in M_\alpha - (\bigcap_{r \neq \alpha} M_r)$ and $n - t_\alpha \in (\bigcap_{r \neq \alpha} M_r) - M_\alpha$. These observations imply that in $D'' = D'_{\{t_\alpha^i\}_{i=1}^\infty}$, $\{M_r D''\}_{r \neq \alpha}$ is the set of maximal ideals. Further $n - t_\alpha \in \bigcap_{r \neq \alpha} M_r D'' = \sqrt{xD''}$ since x is contained in each M_α. Therefore $(n - t_\alpha)^k \in xD''$ for some positive integer k, so that $(n - t_\alpha)^k / x = \xi \in D''$. Consequently, $v_r(\xi) \geqslant 0$ for each M_r containing x, $r \neq \alpha$. And if $x \notin M_\beta \in \Delta$, clearly $v_\beta(\xi) \geqslant 0$. Moreover, $v_\alpha(\xi) = kv_\alpha(n - t_\alpha) - v_\alpha(x) = -v_\alpha(x) < 0$. We then have $\xi \in (\bigcap_{\beta \neq \alpha} D_{M_\beta}) - D_{M_\alpha}$ and (a) is valid.

Remark. That (c) implies (b) in Theorem 2 is a special case of the following more general result:

If A is an invertible ideal of J, an integral domain with identity, if $\{M_\alpha\}$ is the collection of maximal ideals of J containing A, and if $A \supset \bigcup AM_\alpha$, then A has a basis of two elements. In particular, if A is primary for a maximal ideal, A has a basis of two elements.

THEOREM 3. *If D is almost Dedekind and has property (#), D is a Dedekind domain.*

Proof. Let M be maximal in D. By Theorem 2, there exist $u, v \in D$ such that $\sqrt{(u, v)} = M$. Hence (u, v) is M-primary, and therefore a power of M: $(u, v) = M^k$[*3*, p. 813]. Since D is almost Dedekind, (u, v) is invertible and hence M is also invertible. Therefore D is Dedekind as asserted.

Note. Theorem 2, Lemma 7, and Lemma 8 imply that if D is one-dimensional and has property (#), then each nonunit of D is contained in only finitely many maximal ideals of D. Hence the conditions of Lemma 6 also are necessary in order that a one-dimensional Prüfer domain have property (#).

3. EXAMPLES

The first of the following examples exhibits a one-dimensional Prüfer domain with infinitely many prime ideals having property (#) which is not almost Dedekind. The second exhibits an almost Dedekind domain such that all but one of its maximal ideals is the radical of a principal ideal, but such that the domain is not Dedekind.

Example 1. Let A be the domain of all algebraic integers and let $\{p_i\}_{i=1}^{\infty}$ be the sequence of primes of Z. For each i choose a maximal ideal M_i of A lying over $p_i Z$, let $N = A - (\bigcup_{i=1}^{\infty} M_i)$, and let $J_1 = A_N$. A may be expressed as the union of an ascending sequence of Dedekind domains with finite class groups; hence A is one-dimensional and has the QR-property. Consequently, J_1 has these same two properties. It is straightforward to check that $\{M_i J_1\}_{i=1}^{\infty}$ is the set of maximal ideals of J_1 and that each nonunit of J_1 is contained in only finitely many maximal ideals. Hence Lemma 6 shows that J_1 has property (#). But each maximal ideal of A is known to be idempotent, and this property carries over to J_1. Hence J_1 is not almost Dedekind [*3*, p. 814].

Example 2. Denote by Γ the field of rational numbers and let $\{p_i\}_{i=1}^{\infty}$ be the sequence of positive primes in Z. We denote by ω_i a primitive p_ith root of unity for each i, and we let $L = \Gamma(\omega_1, \omega_2, \ldots)$. Nakano showed in [*8*, pp. 426-427] that the integral closure J of Z in L is an almost-Dedekind domain which is not Dedekind. He further shows that given p a fixed prime of Z, there is an ascending sequence $\{L_i\}_{i=1}^{\infty}$ of finite algebraic extensions of Γ such that $L = \bigcup_{i=1}^{\infty} L_i$ and such that the following holds: there exists a fixed positive integer t such that in Z_i, the integral closure of Z in L_i, $pZ_i = (P_{i1} P_{i2} \cdots P_{ik_i})^t$ where the P_{ij} are distinct maximal ideals of Z_i and where P_{ij} decomposes into a product of at least two distinct primes in Z_{i+1} for all

j. These conditions imply that the following construction is possible: Let v be the p-adic valuation of Γ. There exist distinct extensions v_1 and w_1 of v to L_1. We let μ_1 be any extension of w_1 to L. There exist distinct extensions v_2 and w_2 of v_1 to L_2. Let μ_2 be any extension of w_2 to L, etc. Let M_i be the center of μ_i on J and let $M = \bigcup_{i=1}^{\infty} V_i$ where V_i is the center of v_i on Z_i. We let $N = J - (\bigcup_{i=1}^{\infty} M_i)$ and we let $J_2 = J_N$. Since J is almost Dedekind, so is J_2. We next show that $\{MJ_2, M_iJ_2\}$ is the collection of maximal ideals of J_2. Thus suppose Q is a maximal ideal of J such that $Q \subseteq \bigcup_{i=1}^{\infty} M_i$. Then for any j, $Q \cap Z_j \subseteq \bigcup_{i=1}^{\infty} (M_i \cap Z_j)$. For $j > i$, μ_j extends v_i. Hence $\{M_i \cap Z_j\}_{i=1}^{\infty} = \{M_1 \cap Z_j, \cdots, M_{j+1} \cap Z_j\}$, the latter enumeration being into distinct maximal ideals. Thus $Q \cap Z_j = M_r \cap Z_j$ for some r. Since μ_j is the unique extension of w_j to L which is finite on J_2, it is apparent that if for some j, $Q \cap Z_j = M_r \cap Z_j$ where $r < j + 1$, then $Q \cap Z_j = M_r \cap Z_j$ for all j and $Q = (\bigcup_{j=1}^{\infty} Q \cap Z_j) = (\bigcup_{j=1}^{\infty} M_r \cap Z_j) = M_r$. But if $Q \cap Z_j = M_{j+1} \cap Z_j$ for all j, then $Q = \bigcup_{j=1}^{\infty} M_{j+1} \cap Z_j = \bigcup_{j=1}^{\infty} V_j = M$. This proves our assertion concerning the maximal ideals of J_2.

Because $M_i \cap Z_{i+1} \nsubseteq V_{i+1} \cup (\bigcup_{j \neq i} M_j \cap Z_{i+1})$, $M_i \nsubseteq M \cup (\bigcup_{j \neq i} M_j)$. Therefore M_iJ_2 is the radical of a principal ideal for each i. Finally we note that if $x \in M$, then for some j, $x \in V_j$. Hence $x \in M_t$ for all $t > j$. Hence x is contained in infinitely many prime ideals of J_2, and J_2 is not Dedekind.

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